Quantum Spectra and Wave Functions in Terms of Periodic Orbits for Weakly Chaotic Systems

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Special quantum states exist which are quasiclassical quantizations of regions of phase space that are weakly chaotic. In a weakly chaotic region, the orbits are quite regular and remain in the region for some time before escaping and manifesting possible chaotic behavior. Such phase space regions are characterized as being close to periodic orbits of an integrable reference system. The states are often rather striking, and can be concentrated in spatial regions. This leads to possible phenomena. We review some methods we have introduced to characterize such regions and find analytic formulas for the special states and their energies.

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I. SPECIAL QUANTUM STATES

There are many *special quantum eigenstates*, [more precisely *classes* of eigenstates] which often exist in the "quantum chaos" problems that have been considered. These states are associated with special regions of phase space which are 'weakly chaotic'. By this we mean that classical orbits are regular and remain in the region for a sufficiently long time. It may be that almost all the orbits are chaotic, but that is a long time property of little consequence to the formation of quantum states. The regions can be considered as being close to a set of nonisolated periodic orbits. These periodic orbits are not necessarily orbits of the system under study. They could rather be orbits of a 'nearby' integrable reference system.

This paper reviews some recent work by the authors on this topic which is partially published [1], [2], and partially submitted for publication [3], [4]. Some further examples and details will appear in the thesis of Oleg Zaitsev [5]. We have introduced a new technique which is in some ways more general than those previously used. We have also shown how to generalize an older method, the Born-Oppenheimer approximation, to new situations.

Remarkably, only two such kinds of states have been clearly recognized in the literature, but there are numerous others which have been overlooked. This is in spite of the fact that the quantum problems giving rise to these states have been extensively studied numerically. Even in cases where the states dominate the effect under consideration they have not been mentioned. The reason for this oversight is probably as follows: a) the special states are high quantum number states and they are rare in the sense that they are a small fraction of the ordinary states existing in the same energy range, and b) most work has concentrated on energy levels and energy level statistics rather than wavefunctions.

The one special state known to most physicists and engineers, even those not especially concerned with 'quantum chaos', is Lord Rayleigh's *whispering gallery* mode [6]. Although in most respects this case is very well understood, we shall mention a new development here, as well as providing a new [or rather old] easy way to get the main result.

There are many phenomena associated with whispering gallery modes. The whispering gallery in St. Paul's cathedral in London is perhaps the first case to be remarked by a scientist. In 1871, G. B. Airy attempted to explain the phenomenon. Lord Rayleigh, in his *Theory of Sound* of 1894 disagreed with Airy and gave, at the level of ray analysis, the currently accepted explanation. It is amusing that the leading order formula taking the wave nature of sound into account is expressed in terms of Airy functions. The most recent mention of these modes that we have found is by Agam and Altshuler [7]. They have explained the *lack* of whispering gallery modes in the 'Faraday' experiments of Kudrolli, et. al. [8] which shakes a container of water in the shape of a Bunimovich billiard, as due to the fact that significant dissipation occurs at the boundary, precisely where the whispering gallery modes are localized.

The second class of states is familiar to most quantum chaologists if not to the wider physics community. These are the *bouncing ball* modes. There are some phenomena associated with these modes, as well [9].

The most widespread class of 'bouncing ball' states occurs in mixed chaotic systems, that is, systems which have stable periodic orbits as well as unstable ones. Keller and Rubinow [10] considered, in particular, a smooth convex billiard which must in general have a minimum diameter where there is an orbit bouncing perpendicularly to the boundary at two opposing points. Such an orbit is stable and there is a volume of phase space near the orbit which is invariant under the Hamiltonian flow. It's possible to make a harmonic expansion and quantize these states. More generally, there are generalizations of such 'bouncing ball' states to the neighborhood of *any* stable periodic orbit. This requires $\hbar$ to be small enough that the area of the invariant phase space on a surface of section perpendicular to the orbit is of order Planck's constant h.

Although our technique allows an improvement on the harmonic expansion, we shall not consider further such states. Since they typically occur in mixed chaotic systems they have received relatively little attention from the quantum chaos community until recently.

However, there is a rather spectacular generalization of these bouncing ball states to some hard chaotic systems not possessing stable orbits. The best known are the modes of the Bunimovich stadium corresponding to a ball bouncing perpendicularly to the straight sides of the billiard. These were discovered by MacDonald and Kaufman [11] in their pioneering numerical work on that system. Similar modes are known in other billiards with parallel sides [12], [13].

These bouncing ball states have been studied from many points of view, including periodic orbit theory [14], [15]. Paradoxically, much work has been aimed at 'getting rid of them'. This is because the classical bouncing ball periodic orbits dominate the 'trace formula' and the corresponding quantum states dominate the oscillatory part of the density of states, thus obscuring the underlying 'quantum chaos' of the energy level correlations, which was deemed more interesting.

Although these special states are associated with periodic orbits, they are not to be confused with 'scars' [16]. Scarred states are certainly of great interest. They are associated with unstable periodic orbits, and are a relatively weak effect. The special states are associated with a class of reference periodic orbits and are a large effect. They are strongly localized in some sense or another, and for this reason have the potential to lead to distinct phenomena. In fact, sometimes one of a sequence of special states can be regarded as being scarred.

The classical limit of the systems under consideration may be strongly chaotic, or mixed chaos, or pseudointegrable, or even, if you wish, integrable. These are *long time* characterizations of the classical motion, however, and have no direct correlation with quantum states. To support a special state, the system must be weakly chaotic in the sense that there is region of phase space where the classical motion is nearly integrable for a short but long enough time. The size of this region and the characteristic times are functions of $\hbar$. There is also a condition on the way that orbits escape from this region.

II. SOME EXAMPLES

Here are some examples of systems with special states which have not been remarked in the literature. We consider only two dimensional billiards, for simplicity. We begin with three variations on the Bunimovich stadium, which has horizontal straight parallel sides of length $2a$ capped by semicircles of radius R, where a/R is of order unity. We consider the solutions of the Helmholtz equation $\left(\nabla^2 + k^2\right)\Psi(x,y) = 0$, where Ψ vanishes on the boundary for values of k^2 on the spectrum. We consider only cases where k is large, but not too large. That is, we do not automatically take the limit $k \to \infty$ since the states of interest will be of negligible number in that limit. This is equivalent to considering $\hbar$ to be small.

A. Bunimovich stadium

First, we review the original Bunimovich stadium which has a set of *nonisolated* bouncing ball orbits, i.e. those orbits bouncing perpendicularly between the straight sides. Such a set of orbits gives an especially large contribution to the Gutzwiller trace formula, as is well known and we will remind you below. This leads to the result that there is a very strong *correlation of energy levels.* In fact, the energy level correlation is dominated by the bouncing ball *quantum states.* These are states of the approximate *Born-Oppenheimer* form,

$$\Psi_{nm}(x,y) = \Phi_n(y|x)\psi_m(x), \tag{1}$$

where

$$\Phi_n(y|x) = \sin n\pi \frac{y + R - \xi_B(x)}{2\left(R - \xi_B(x)\right)}. \tag{2}$$

[The Born-Oppenheimer treatment of the Bunimovich billiard was first given in reference [17] and is discussed further in reference [12].] Here $\xi_B(x) = 0$, $|x| \le a$, $\xi_B(x) = R - \sqrt{R^2 - (|x|-a)^2} \approx \frac{1}{2}(|x|-a)^2/R$ for $|x| \ge a$, $(|x|-a) << R$. This makes $\Phi(y|x)$ vanish when y is on the billiard boundary. Treating x as a slowly varying parameter in $\Phi(y|x)$, it is seen that $\psi_m(x)$ satisfies

$$-\psi_m''(x) + V(x)\psi(x) = E_m\psi(x) \tag{3}$$

where $V(x)$ is a sort of square well potential,

$$V(x) \approx n^2\pi^2\xi_B(x)/2R^3 \tag{4}$$

and the energy

$$k^2 = k_{n,m}^2 \approx (n\pi/2R)^2 + E_m. \tag{5}$$

Thus, for large n, $\psi_m(x)$ is something like $\sin(m\pi(x+a)/2a)$, with $m << n$. Therefore $E_m \approx (m\pi/2a)^2$.

The leading term of the energy, $(n\pi/2R)^2$, is a quantization of the bouncing ball orbits. The next term, is a quantization of the transverse motion, motion in the potential $V(x)$. If this quantization is carried out semiclassically, it too is expressed in terms of the periodic orbits of motion in the potential.

These states have wavenumbers $k_{n,m} \approx n\pi/2R + O(1)$. This corresponds to the leading term in the trace formula, coming from the nonisolated bouncing ball orbits, which varies as $\exp(4ikR)$. The length $s_1 = 4R$ satisfies, approximately, $\exp(is_1(k_{n+N,m} - k_{n,m})) = 1$. It is the leading element of the *length spectrum.* The next element, $s_2 = 8R$ also satisfies this condition. In the trace formula it comes from once repeated bouncing ball orbits. Thus the bouncing ball states account the most of the correlations in the length spectrum which are multiples of $4R$.

We show in Fig. 1 a density plot of $\Psi_{10,1}$, as given by Eq. (1). [Actually, in most cases we show the absolute square of the theoretical wavefunction.] The wavefunctions we show are theoretical. Later we make some remarks on the accuracy of the theoretical predictions.

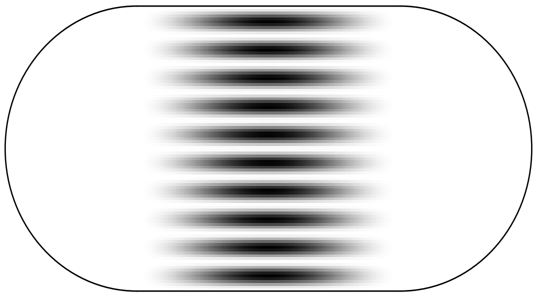

FIG. 1. Bouncing ball mode of the standard Bunimovich stadium. Straight side has length $2a$, endcap radius is $R = a$. The quantum numbers are $n = 10$, $m = 1$. Figs. 1,2,3,4,6,7 give the stadium shape and a density plot of the square of the theoretiacl wavefunction.

It is, of course, not true that the bouncing ball orbits by themselves give the bouncing ball states [14]. These orbits occupy a set of measure zero in phase space. Nevertheless, the most natural way to characterize the phase space which supports the states is that it is in a certain neighborhood of the bouncing ball periodic orbits.

B. Slightly sloped stadium

The first variation is the slightly sloped stadium considered by Primak and Smilansky [15]. In this case the end caps have radius $R(1 \pm \epsilon)$ and the sides, instead of being horizontal, have slopes $\pm\epsilon R/a$. The parameter ϵ is small. [To minimize diffraction, we suppose that the sides merge smoothly to the end caps with no change of slope. The results are given to leading order in ϵ only.] In this case, the special states are *strongly modified* even if ϵ is quite small.

In fact, the formulas above hold upon replacing $\xi_B \to \xi_S$ where the major new feature is that, for $|x| < a$,

$$\xi_S \approx -\epsilon x R/a. \tag{6}$$

This gives for $V(x)$ a steep sided well with a sloping bottom,

$$V(x) \approx -\epsilon n^2 \pi^2 x/2aR^2, \tag{7}$$

for $|x| < a$. The condition that the slope be large enough to change substantially the states from the Bunimovich, $\epsilon = 0$, case is $\epsilon n^2/R^2 \geq 1/a^2$. If this inequality is strong, the small m states ψ_m are approximately Airy functions concentrated near $x = a$. In Fig. 2 we show the state $\Psi_{10,2}$ for $\epsilon = 0.05$, $a = R$.

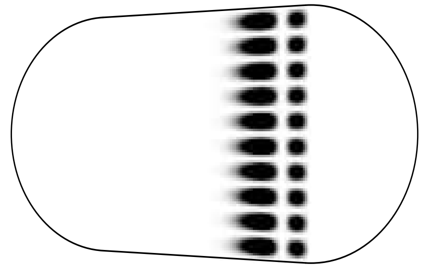

FIG. 2. Special mode of a slightly sloped Bunimovich stadium. The slope parameter is $\epsilon = 0.05$, and $R = a$. The quantum numbers are $n = 10, m = 2$.

One motivation for considering this case is that the nonisolated periodic orbits mathematically disappear for any finite ϵ no matter how small. However, if ϵ is small, there is a region of phase space which remains close to the bouncing ball orbits of the original stadium. Thus, there are still special states related to the bouncing ball orbits of the standard stadium. However, it is complicated and not very enlightening to describe these states in terms of the periodic orbits of the slightly sloped stadium itself.

There is a parametrically different dependence on ϵ of the states as compared with the energy correlations. We see below that if $kR\epsilon << 1$, the leading terms of the trace formula, or in other words, the energy correlations due to the bouncing ball states, are little modified [15]. This is the natural expectation, since the condition is simply that the wavelength is large compared with the change of radius of the endcap. On the other hand, we have just found that if $ka\sqrt{\epsilon} \geq 1$ the eigenstates and energies are substantially changed. Thus, if $(ka)^{-2} << \epsilon << (kR)^{-1}$, the states are changed but the correlations are not.

C. Baseball stadium

As ϵ in the previous example becomes of order unity, the special state again changes. Note that for finite ϵ, there is a set of nonisolated periodic orbits through the center of the larger circle, whose angular range is of order ϵ. If the endcaps have radii R_2, R_1 with $0 < R_2 - R_1 < 2a$, the angular range of such periodic orbits is $\pm\theta_a$ measured away from the vertical, where $\sin\theta_a = (R_2 - R_1)/2a$. This region supports special states $\Psi_{n,m}(r, \theta)$, where r, θ are polar coordinates for the large circle.

The Born-Oppenheimer approximation used above relies on having a *fast* variable and a *slow* variable. The ansatz, Eq. (1), was made with y the fast variable and x the slow. Fast and slow mean from the classical point of view that the motion in the y direction is much faster than in the x direction. From a wave viewpoint it means that $|\partial\Phi/\partial y| >> |\partial\Phi/\partial x|$ which allows the x dependence of Φ to be treated parametrically.

The states of this section would seem to have a fast radial motion and a slow angular motion. However, close to the center of the circle this separation is not valid.

Nevertheless, the Born-Oppenheimer approximate state gives the result for the outer region, and that suffices.

Thus, the states are approximated as above, replacing the fast coordinate $y \to r$ and the slow coordinate $x \to \theta$. Then

$$\Phi(r|\theta) \approx \frac{1}{\sqrt{kr}} \cos\left(kr + \frac{\nu(\theta)^2 - \frac{1}{4}}{2kr} + \alpha(\theta)\right) \quad (8)$$

for kr large compared with ν. This function is an asymptotic Bessel function of order ν, [a superposition of Bessel and Neumann functions] and $\psi(\theta)$ satisfies $\psi''(\theta) + \nu(\theta)^2\psi(\theta) = 0$. The billiard boundary is $r_\partial(\theta) = R_2 + \xi_C(\theta)$ and Φ must vanish for $r = r_\partial(\theta)$. Here $\xi_C = 0$, $-\theta_a < \theta < \pi + \theta_a$ and $\xi_C = R_2(1/\cos(\theta+\theta_a) - 1) \approx \frac{1}{2}R_2(\theta+\theta_a)^2$ for $\theta < -\theta_a$ and $(\theta+\theta_a)^2 << 1$, with a similar formula near $\theta = \pi+\theta_a$. It can be shown that $\alpha = \frac{1}{2}k(\xi_C(\theta+\pi) - \xi_C(\theta)) \pm \pi/4$ and $\nu^2 = E_m - V(\theta)$ with $V(\theta) = k^2R_2(\xi_C(\theta+\pi) + \xi_C(\theta))$. For large enough k there is an approximate solution $\psi(\theta) = \sin(m\pi(\theta+\theta_a)/2\theta_a)$, $|\theta| < \theta_a$, $\psi(\theta)$ vanishes outside this region for $|\theta| < \pi$, and $\psi(\theta) = \pm\psi(\theta+\pi)$. For this solution $E_m = (m\pi/2\theta_a)^2$. The energy parameter k solves $kR_2 + E_m/kR_2 = (n \mp \frac{1}{4})\pi$. For large n this has the approximate solution $k_{n,m}^2 = \left((n \mp \frac{1}{4})\pi/R_2\right)^2 - E_m/R_2^2$. There are $2n - \frac{1}{2} \mp \frac{1}{2}$ radial nodes.

We show this state for $n = 10$, $m = 1$, in Fig. 3. Notice some changes of sign as compared with the bouncing ball between the straight sides. In particular, a deviation of the billiard sides in the direction of *narrowing* the channel gives a repulsive effective potential V, thus containing the wave function. For a billiard which is a deviation from a circle, an outward deviation gives the repulsive effective potential.

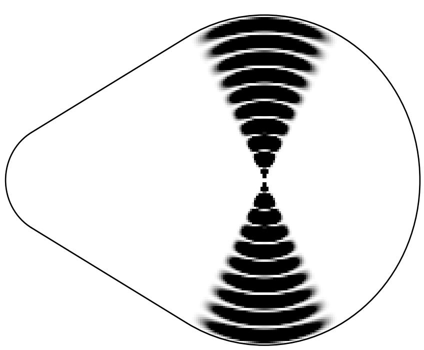

FIG. 3. Special mode of a ‘baseball’ stadium. $R_2 = 1.5a$, $R_1 = 0.5a$. Quantum numbers are $n = 10$, $m = 1$. The theoretical wave function in this and the next figure is inaccurate near the center of the circle.

D. Almost circular stadium

The above Eq. (8) can be applied to find the low angular momentum states of any almost circular billiard. Energy levels and wavefunction statistics were extensively studied in this system [24]. Take $r_\partial = R + \xi_D(\theta)$ to describe the billiard, with $\xi_D(\theta)$ small. For the Bunimovich stadium with side $a << R$, $\xi_D \approx a\,|\cos\theta|$. Polar coordinates at the center of the billiard are used and approximations neglecting $(a/R)^2$ have been made.

In this case the potential $V(\theta) = 2k^2aR\,|\cos\theta|$, an attractive triangular well near $\theta = \pm\frac{1}{2}\pi$. It is periodic with period π. For large k^2aR, the lowest eigenstates will be concentrated near $\theta = \pm\frac{1}{2}\pi$. We show the state $n = 10$, $m = 2$ in Fig. 4.

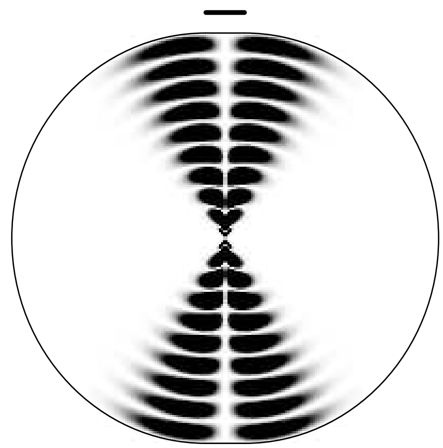

FIG. 4. Special low angular momentum mode in a Bunimovich stadium with a short side, $a = 0.1R$. Quantum numbers are $n = 10$, $m = 2$.

All of these variations on the Bunimovich stadium have been considered in the literature, except for the baseball stadium. The only special states previously pointed out, however, were for the Bunimovich stadium itself.

III. FINDING AND APPROXIMATING SPECIAL STATES

A. History

The whispering gallery modes and the bouncing ball modes corresponding to stable periodic orbits were first quantized by Keller and Rubinow [10] by what is sometimes called the *ray method.* This technique starts with the classical mechanics or rays, and exploits caustics and adiabatic invariants. However, the assumptions usually made are unnecessarily strong, and these special states still exist even if the caustics are only a short time approximation.

Another group of methods starts from the partial differential equation, the Helmholtz equation in the case of billiards. The *parabolic equation method* invented independently by Leontovich [18] and by Fock [19] chooses coordinates astutely, and finds appropriate scale factors, allowing an approximation to the partial differential equation. It often relies on the ray method to moti-

vate the manipulation of the PDE. The *etalon* method of Babich and Buldyrev [20] is an improvement of this which involves choosing a characteristic example, or 'etalon', which captures the essential features of the given problem. The *extended Born-Oppenheimer method* [EBO] has been used in this context only for bouncing ball states between parallel sides of a billiard. We showed above a couple of generalizations of this technique. Other generalizations are given elsewhere [4].

The last three methods are closely related. They use different language and motivation and make approximations in a different order, but the main result is the same. Systematic corrections to the leading result have been extensively studied for certain examples, and these corrections are apparently of a different form for the different methods. We prefer the EBO when it applies since it is simpler and better known than the other techniques.

B. Bogomolny operator

We also have introduced a technique based on Bogomolny's *surface of section transfer operator* [21] which in certain ways is more general than the above methods. It also has the advantage that the transfer operator T is closely related to the trace formulas and to important 're-summations' of the trace formulas. In addition, it makes somewhat more precise the notion of a region of phase space which is nearly integrable for short times.

The operator $T(s,s'|E)$ introduced by Bogomolny is a generalization of the *boundary integral method* used to obtain numerical solutions for billiard problems. The main equation of the boundary integral method is derived using Green's theorem. It takes the form

$$\mu(s) = \int_\partial ds' K(s,s'|E)\mu(s'), \tag{9}$$

a Fredholm integral equation. The kernel or operator K depends parametrically on the energy E. Eq. (9) has a nontrivial solution only when E is on the spectrum. For Dirichlet conditions, $\mu(s) = \partial\Psi(\mathbf{r})/\partial n$, the normal derivative of the wavefunction evaluated at the position on the boundary labelled by s the distance along the perimeter. The integral in Eq. (9) is over this boundary.

The boundary coordinate s together with its conjugate momentum, the momentum parallel to the boundary p_s, gives the Birkhoff coordinates for a surface of section of a billiard. Bogomolny's method essentially approximates $K(s,s'|E)$ by its asymptotic form, which we call $T(s,s'|E)$. It generalizes to a broad class of surfaces of section, not restricted to billiards, for which there are often also exact kernels. For a billiard, in Birkhoff coordinates,

$$T(s,s'|E) = \left(\frac{k\left|\partial^2 L(s,s')/\partial s\partial s'\right|}{2\pi i}\right)^{\frac{1}{2}} e^{i\left(kL(s,s')+\mu\right)}, \tag{10}$$

where L is the distance between boundary points, $k = \sqrt{2mE}/\hbar$ and $\mu = \pi$ for Dirichlet conditions. We shall suppress μ and write the prefactor as (.) for simplicity. The essential variation of T is given by the exponential dependence on $kL = \hbar kL/\hbar = S(s,s'|E)/\hbar$,where S is the action of the classical straight line orbit between boundary points s, s'.

Because $\hbar$ is supposed to be small compared with typical classical actions the exponential is rapidly varying, and the method of stationary phase usually applies to integrals in which T appears. We deal always with billiard examples for which it is convenient to take $\hbar = 2m = 1$, and kL is typically large.

The action S generates the classical *surface of section map* from

$$p_s = \partial S/\partial s,\ p_{s'} = -\partial S/\partial s'. \tag{11}$$

Composition of powers of T have intermediate points of stationary phase determined by this map so in effect, longer and longer orbits can be built up by iteration of the T operator.

Corresponding to Eq. (9) is the fundamental equation

$$\psi(s) = \int ds' T(s,s'|E)\psi(s'). \tag{12}$$

Our new method, for the cases corresponding to special states, solves this equation directly and quasiclassically, that is, it exploits the stationary phase approximation to do the integral.

Eq. (12) has solutions only for E values on the quasiclassical spectrum such that

$$D(E) = \det(1 - T(E)) = 0 \tag{13}$$

This Fredholm determinant is a well defined version of the *dynamical zeta function* [22], a resummation of the trace formula [23]. The trace formula itself is given by

$$\begin{aligned} d_{osc}(E) &= \frac{-1}{\pi}\mathrm{Im}\frac{\mathrm{d}\ln\mathrm{D(E)}}{\mathrm{dE}} \\ &= \frac{-1}{\pi}\mathrm{Im}\frac{\mathrm{dTr}\ln\left(1-\mathrm{T(E)}\right)}{\mathrm{dE}} \\ &= \frac{-1}{\pi}\mathrm{Im}\frac{\mathrm{d}}{\mathrm{dE}}\sum_{\mathrm{r}}\frac{1}{\mathrm{r}}\mathrm{TrT^r} \end{aligned} \tag{14}$$

Here the density of states, $d(E)$, is expressed as

$$d(E) = \sum_a \delta(E - E_a) = d_{Weyl}(E) + d_{osc}(E) \tag{15}$$

where d_{Weyl} is the smoothed density of states. The traces of powers of T are expressed in terms of periodic orbits when the integrals are done in stationary phase. In the case of the special states, as well as the integrable case, however, only $r-1$ integrals are well approximated by that method.

C. Special states and periodic orbits

We take as an example the slightly sloped billiard described above. Introduce a reference billiard system consisting of a channel of width $2R$, i.e. with boundaries at $y = \pm R$. The actual orbits are compared with the bouncing ball orbits of this channel.

We want to choose a surface of section such that the nominal periodic orbits are described by a diagonal element, $T(x, x)$. Such a surface of section is the upper half of the billiard. It is convenient to label position along it by the corresponding position x on the upper side of the reference billiard. [Note that a simple change of coordinates, e.g. $s \to s(x)$ does not affect the result.]

We now approximate the actual $T(x, x')$ in a way valid for $|x - x'| << R$. Thus, we may expand

$$L(x, x') \approx 4R + \frac{1}{2R}\left(x - x'\right)^2 - \xi_S(x) - \xi_S(x') \qquad (16)$$

where ξ_S, given by Eq. (6) above, is small except in the endcaps. The first two terms in this formula are an approximation to the perfect channel, and ξ_S approximates the difference between the sloped billiard and the channel.

Assuming for the moment that $k\xi_S(x) << 1$, we can approximate

$$\begin{aligned} T(x, x') &\approx (.)e^{ikR\left(4+\frac{1}{2}\left(\frac{x-x'}{R}\right)^2\right)} \\ &\quad \times e^{-ik\left(\xi_S(x)+\xi_S(x')\right)} &(17)\\ &\approx (.)e^{ikR\left(4+\frac{1}{2}\left(\frac{x-x'}{R}\right)^2\right)} \\ &\quad \times \left(1 - ik\left(\xi_S(x) + \xi_S(x')\right)\right). &(18) \end{aligned}$$

This T operator has a form that allows direct solution of Eq. (12). Next assume that the solution $\psi(x)$ of Eq. (12) is slowly varying and expand $\psi(x') \approx \psi(x) + (x' - x)\psi'(x) + \frac{1}{2}(x' - x)^2 \psi''(x)$. The rapidly varying exponential in T ensures that the important contributions to the x' integral lie close to x, according to the estimate $(x' - x)^2 \sim R/k$.

One may carry out the integral of Eq. (12) and obtain the equation $\psi(x) = \exp(i(4kR - E_m R/2k))\psi(x)$ with the condition that ψ satisfies the Schrödinger equation (3) with potential V of Eq. (7), where the replacement $n\pi \to 2kR$ is made. The energy eigenvalues $k_{n,m}^2$ are found by insisting that the exponential is unity. Thus,

$$4k_{n,m}R - E_m R/2k_{n,m} = 2\pi n \qquad (19)$$

Since $k\xi_S$ becomes large as x goes into the endcap region, the approximation of Eq. (18) seems to break down. However, the wave function becomes small in that region. As long as the approximate wave function continues to be small, the approximation is satisfactory.

It is also possible to extend this result to a parameter regime such that $k\xi_S$ becomes of order unity or larger [1]. It is only necessary that $k\xi_S$ be slowly varying compared with the leading term.

D. Nearly integrable phase space region

The above formulation can be regarded as defining a phase space region where the system is nearly integrable. The two arguments of T together with the surface of section map give an area on the 2-dimensional phase space such that T is well approximated by Eq. (18). This, in turn, is the T operator for an integrable system, characterized by being a function of the coordinate difference, together with a leading correction. For the case discussed, the surface of section phase space is approximately $|x| \le a$, $|p| \le \hbar k \left|x - x'\right| / R \approx \hbar\sqrt{k/R}$. The two-dimensional surface of section phase space region defines a four dimensional phase space region, from which the orbits only slowly escape.

A classical orbit at a typical point in this phase space region is not too different from a nearby classical orbit of the ideal integrable region. Of course, continuing this orbit for long times allows it to escape from the region. In the case at hand, almost all such orbits are chaotic with positive Lyapunov exponents.

However, if the fundamental equation $T\psi = \psi$ has a solution $\psi(x)$ which is large only in the nearly integrable region, there is no need to consider the very long orbits. That gives another condition on the perturbation. In terms of the effective potential, $V(x)$, it means that V must become repulsive outside the region. The sign of V depends on the sign of the perturbation, *and* on the sign of the quadratic term in the integrable part of T. Thus, for bouncing ball states between parallel sides, a perturbation shortening the bounce path is repulsive, while for radial bounces between circular sides, a perturbation lengthening the path is repulsive.

E. Trace formula

Consider the contribution to the trace formula from the period 1 orbits, of Eq. (15), $d_{osc,1} = \frac{-1}{\pi}\mathrm{Im}\frac{\mathrm{d}}{\mathrm{dE}}\mathrm{TrT}$. In our case, $\mathrm{TrT} = \int \mathrm{dxT(x,x)}$. The hard chaos case usually considered assumes that this integral can be done by stationary phase, with contributions only from the neighborhood of one or more points. Using Eq. (17) we see that the stationary phase approximation is not good unless $\epsilon ka >> 1$. In the case of the opposite strong inequality, $\epsilon ka << 1$ the x integral gives just a factor of $2a$, to leading approximation. However, we saw that if $\sqrt{\epsilon}ka > 1$, there are strong effects on the wave functions. Thus, although the wave functions are sensitive to the slope, the energy level correlations are not in this parameter regime.

This is a fairly general result. Namely, special states are quite sensitive to perturbations. The energy level correlations are much less sensitive to the same perturbation.

IV. MORE EXAMPLES

We briefly give a few more examples. The problems already mentioned can be solved by the Born-Oppenheimer method. In the examples of this section, the Born-Oppenheimer method is more limited or more cumbersome, or impossible to use, but the Bogomolny operator method succeeds.

First, the whispering gallery modes can easily be studied. These modes are concentrated near the boundary of a sufficiently smooth and sufficiently convex billiard. The case in which the corresponding classical orbits are also confined near the boundary is the only one studied in detail in the literature. Assuming a smooth enough convex billiard with everywhere positive curvature on the boundary, the appropriate coordinates are s, the distance along the perimeter, and ρ, the distance from the perimeter toward the center of curvature at s. Let $R(s)$ be the radius of curvature at point s. The Born-Oppenheimer ansatz is $\Psi(r|s) = e^{iks}\Phi(\rho|s)\psi(s)$, $\Phi(\rho|s) = C(s)J_{\nu(s)}(k(R(s)-\rho))$ and $\psi(s) = e^{-ikR(s)f(s)}$. Here J_ν is a Bessel function of the first kind, with variable index ν and $C(s)$ is a slowly varying prefactor, determined by normalization.

The whispering gallery modes are characterized by large ν, slightly less than kR. It turns out that $\nu = kR(1-f)$ where f is small. One choice is using the Born-Oppenheimer approach, treating the s variation as slow compared to ρ, [except for the explicit factor e^{iks}], and determining $\nu(s)$ by the condition $J_{\nu(s)}(kR(s)) = 0$. [A standard asymptotic formula expresses J_ν in terms of an Airy function.] Alternatively, ψ can be determined as the solution of $T\psi = \psi$ and Φ can be obtained from that. The result is $f(s) = z_m/2^{\frac{1}{3}}(kR(s))^{\frac{2}{3}}$, where z_m is a zero of the Airy function.

It turns out that the condition $|\partial\Phi(\rho|s)/\partial\rho| >> |\partial\Phi(\rho|s)/\partial s|$ that the Born-Oppenheimer approximation is valid is the same as the classical condition for a caustic. [The quantizing caustic would be at approximately $\rho = R(s)f(s)$.] The method using the T operator can be generalized, however, to study cases where there are points of vanishing curvature on the boundary and caustics do not exist. Associated with a caustic is an adiabatic invariant. As an orbit passes a zero curvature point, the adiabatic invariant jumps to a new value. The T operator can account for this, but the Born-Oppenheimer wave function cannot be used. We show in Fig. 5 a whispering gallery state in a stadium billiard with small short side $a = 0.05R$. There are no caustics in this case.

A second example is the slightly sloped trapezoidal billiard of Morse and Feshbach [25], which was recently reconsidered by Kaplan and Heller [26]. The remarkable states shown in the next two figures were not mentioned, however. As shown in Fig. 6, this billiard is almost a square, with vertical sides, say, $x = 0$, $x = 1$, and horizontal sides $y = 1$, $y = \epsilon x$, where $0 < \epsilon << 1$. The Born-Oppenheimer method may be used to find states concentrated near $x = 0$, similar to the states of the slightly sloped stadium. However, there are states associated with the $(1,1)$ period orbits of the unperturbed square. These orbits are rectangles making a 45° angle with the $x-$axis. This may be solved using the Bogomolny technique by extending the billiard antiperiodically in the x direction, and using as surface of section the upper boundary, $y = 1$. The effective potential is $V(x) \propto \epsilon k^2 |x|$, $-1 \leq x \leq 1$, repeated with period 2. States of this extended problem are superimposed to find a solution. A typical case is shown in Fig. 7.

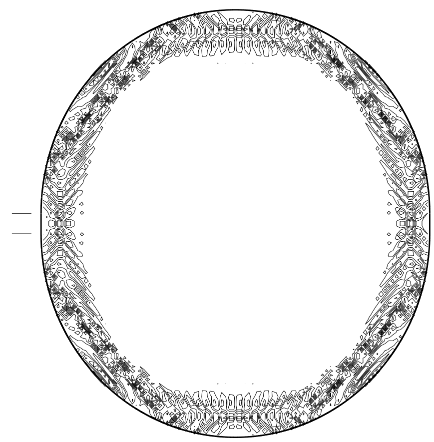

FIG. 5. Contour plot of a numerically obtained whispering gallery mode in a stadium billiard, $a = 0.05R$. There are no caustics in this case, and almost all classical orbits circulating near the boundary eventually escape. The distance between the short parallel lines is the length of the straight side. The wavenumber is $k = 242.7611/R$.

A last example is again the almost circular stadium. We now study states near a higher period orbit. In Fig. 8 we show a numerical state near the (1,4) almost square orbits in a stadium with side $a/R = 0.01$. We have not found any convenient coordinates which we can classify as fast and slow, so the EBO does not work. The $T-$operator approach is straightforward, but a little complicated [1]. In the $T-$ operator approximation, there are two nearly degenerate states. That is, the T operator has inequivalent solutions of the same energy corresponding to orbits moving clockwise or counterclockwise. These states do not couple in the T operator approximation, and the appropriate symmetric combinations have nearly the same energy.

From this figure it is seen that the states are made up of waves along the rays of classical angular momentum $l \approx \hbar k/\sqrt{2}$, i.e. along straight lines whose closest approach to the center is $1/\sqrt{2}$. A perfect circle would have a caustic of this radius. Because the stadium is chaotic, such a caustic does not exist, except as a short time approximation.

V. SUMMARY

We have reviewed some of the salient features of special classes of states which often occur in the systems of interest to quantum chaologists. Perhaps these cases

are common because there is a tendency to construct billiards of straight lines and circles. The system as a whole may be chaotic, in other words, almost all orbits of the system may have positive Lyapunov exponents. However, for some shorter time, a subset of phase space may be close to that of a set of nonisolated periodic orbits of some integrable reference system, and this leads to the special states.

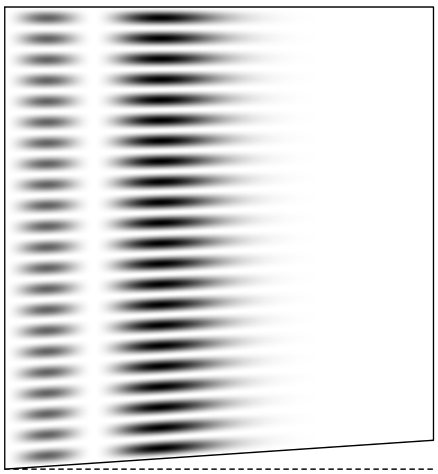

FIG. 6. State along the long side of a nearly square trapezoid. Slope of bottom is $\epsilon = 1/16$. Quantum numbers are $n = 22$, $m = 2$. In this figure and the next, the dashed line is the $x-$axis.

These special states have rather striking wavefunctions. As a result, there are phenomena and even possible applications associated with them. In particular, the whispering gallery modes have long been known to give rise to interesting effects. One standard 'application' is that special states often appear in weakly perturbed integrable systems. If these states are to be avoided, a regular resonant cavity must be constructed much more precisely than the condition $\delta x << \lambda$, where δx is the deviation from the ideal. Also, because there are a sequence of special states regularly spaced in energy, the special states often numerically dominate the trace formula.

Typically, there are two or more scales of variation in connection with the special states which can be identified. If coordinates can be found such that one coordinate is fast and the other is slow, standard adiabatic approximations, such as the Born-Oppenheimer approximation, can be used.

Bogomolny introduced a surface of section transfer operator some time ago as a means of studying the spectrum, i.e. the trace formula. We have shown how this operator can also used to find the special eigenstates. The technique works if the operator can be approximated as a rapidly varying integrable part, and a more slowly varying correction to it. It is in some ways more general than the other methods.

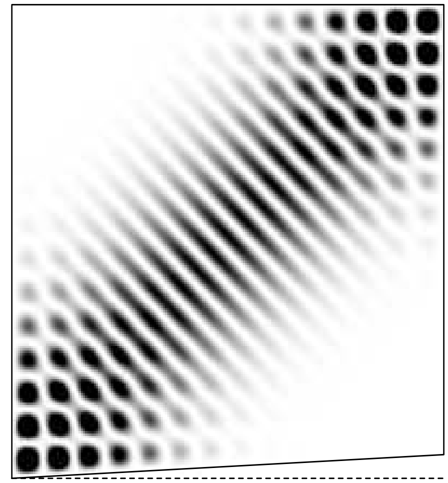

FIG. 7. State along the diagonal of the nearly square trapezoid of Fig. 6. Quantum numbers are $n = 27$, $m = 1$.

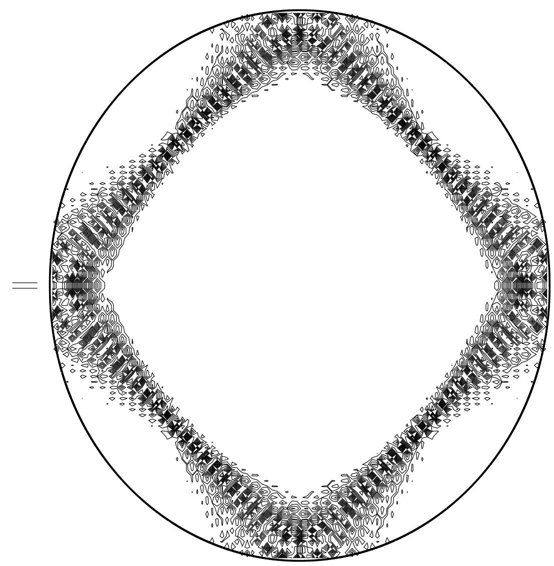

FIG. 8. A contour plot of a numerically obtained state near the (1,4) periodic orbits in a Bunimovich stadium billiard with a short straight side, $a/R = 0.01$. The wavelength of the state is $\lambda = 2.23969a$. The parallel lines show the length of the short side, $2a$.

The special states are typically rather rare, in the sense that they are a small fraction of all the states in a given (high) energy range. In leading approximation, they do not couple to the other states. The energies of the states are predicted to good approximation, relative to the energy spacing of the given class of states, and even better absolutely. However, the accuracy is not necessarily good compared with the mean level spacing of all the levels. Further, it may happen that 'accidentally' there is a nonspecial state with energy very close to that of the special state. Then terms neglected in our approximation can mix these states. Many phenomena are independent of such mixing, however.

In this article we gave a number of examples of such special states, several of which appear for the first time in print. We hope our pictures will tempt an experimentalist to find some of these states in one of the several systems, water trays, acoustic plates, microwave and laser cavities, optical fibers, quantum dots, ..., to which the theory applies.

VI. ACKNOWLEDGMENTS

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